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Investigation of some problems on the generalized frames

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Abstract. Frame theory is an interesting topic that has been studied in both abstract and applied aspects. In recent years, we have several articles on generalized frames, because they sometimes are more flexible than ordinary frames. This paper investigates some of the problems with the conditions that make a sequence a frame, as a generalized form of frames.
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1. Introduction

Every element in a Hilbert space can be represented as a linear combination of the elements of a frame, but this representation is not necessarily unique. This fact is essential in the application. Moreover, K-frames were introduced as special generalizations of frames [5]. Furthermore, controlled frames also have been introduced and investigated [3]. These generalized frames may provide more flexibility in some approaches. Frame theory also plays a foundational role in signal processing, image processing, data compression, sampling theory, and more; It is also productive for researching abstract mathematics.

In this paper, an exercise and a proposition of [4] are examined and then extended to K frames and controlled frames due to their attractive and flexible features.

Before proceeding the main results, some important definitions are provided in the following.

A sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ is called a Bessel sequence if there exists $B > 0$ such that

$$\sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\,\|f\|^2 \qquad (f \in \mathcal{H}).$$

By [4, Theorem 3.2.3], $\{f_n\}$ is a Bessel sequence if and only if the mapping, $T : \ell^2 \to \mathcal{H}$, $T(\{c_n\}_{n=1}^{\infty}) := \sum_{n=1}^{\infty} c_n f_n$ is a well-defined bounded linear operator with $\|T\| \leq \sqrt{B}$. Then T is called synthesis operator, related to $\{f_n\}_{n=1}^{\infty}$. Also, the adjoint operator of T is given by $T^* : \mathcal{H} \longrightarrow \ell^2$, $T^*(f) = \{\langle f, f_n\rangle\}_{n=1}^{\infty}$ and is called analysis operator for $\{f_n\}_{n=1}^{\infty}$. Moreover, a sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ is called a frame if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2 \qquad (f \in \mathcal{H}).$$

The constants A and B are called lower and upper frame bounds, which are not unique. The optimal lower frame bound (resp. the optimal upper frame bound) is the supremum over all lower frame bounds, (resp. the infimum over all upper frame bounds).

The frame operator is defined as $S : \mathcal{H} \longrightarrow \mathcal{H}$, $\quad S(f) = TT^*(f) = \sum_{n=1}^{\infty} \langle f, f_n\rangle f_n$,

Now suppose that $K \in \mathcal{B}(\mathcal{H})$. The range of K is denoted by $R(K)$. A sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ is called a K-frame for $\mathcal{H}$, if there exist constants $A, B > 0$ such that

$$A\|K^* f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2 \qquad (f \in \mathcal{H}).$$

The constants A and B are called lower and upper K-frame bounds of $\{f_n\}_{n=1}^{\infty}$, respectively. The synthesis operator, analysis operator and frame operator of K-frames are similarly defined to frames. Note that the frame operator S of K-frames is not necessarily invertible. However, S is invertible on $R(K)$, whenever K has closed range.

Moreover, let $GL(\mathcal{H})$ denotes the set of all bounded linear operators which have bounded inverses. Following [3], a countable family of vectors $\Psi = \{\psi_n\}_{n=1}^{\infty}$ is controlled by the operator $C \in GL(H)$, or is called a C-controlled frame if there exist two constants $0 < m_{C\Psi} \leq M_{C\Psi} < \infty$, such that

$$m_{C\Psi}\|f\|^2 \leq \sum_{n=1}^{\infty}\langle f, \psi_n\rangle\langle C\psi_n, f\rangle \leq M_{C\Psi}\|f\|^2, \quad (f \in \mathcal{H}).$$

The controlled frame operator for frame Ψ is defined as $L_{C\Psi}f = \sum_{n=1}^{\infty}\langle f, \psi_n\rangle C\psi_n,\ (f \in \mathcal{H})$. For more details about generalized frames we refer to [[1], [2], [3], [6].] Finally, we finish this part by [lemma3.2.6, [4]], which says if $\{f_k\}_{k=1}^{\infty}$ is a sequence of elements in Hilbert space $\mathcal{H}$ and there exists constant B > 0 such that the condition of Bessel sequence there holds for all f in a dense subset V of $\mathcal{H}$. Then $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence for $\mathcal{H}$ with the same bound.

2. Main results

In [Lemma 5.1.9 [4]], author shows that it is enough to check the frame condition on a dense set, i.e., if $\{f_k\}_{k=1}^{\infty}$ is a sequence of elements in Hilbert space $\mathcal{H}$ and there exist constants A, B>0 such that the condition of frame there holds for all f in a dense subset V of $\mathcal{H}$. Then $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with the same bounds.

Now there is a generalization of this lemma to K-frames.

Proposition 2.1. Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of elements in $\mathcal{H}$ and there exists constants $A, B > 0$ such that $A\|K^*f\|^2 \leq \sum_{n=1}^{\infty}|\langle f, f_k\rangle|^2 \leq B\|f\|^2$ for all f in a dense subset V of $\mathcal{H}$. Then $\{f_k\}_{k=1}^{\infty}$ is a K-frame for $\mathcal{H}$ with bounds A, B.

Proof. By [lemma 3.2.6, [4]], $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence. So, the right hand of K-frame inequality holds for all elements in $\mathcal{H}$. Now let g $\in (H = \overline{V}) - V$ such that

$$\|T^*g\|^2 < A\|K^*g\|^2.$$

By density of V in $\mathcal{H}$ there exist a sequence $\{f_n\}_{n=1}^{\infty} \subseteq V$ such that $\lim_{n\to\infty} f_n = g$ Then, Since $T, K \in B(H)$, We have $\lim_{n\to\infty}\|T^*f_n\|^2 < A\lim_{n\to\infty}\|K^*f_n\|^2$. Thus, there exists $N \in \mathbb{N}$ such that $\|T^*f_N\|^2 < A\|K^*f_N\|^2$. This is contradiction. □

Now we want to extend this proposition to controlled frames.

Proposition 2.2. Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of elements in $\mathcal{H}$ and there exists constants $A, B > 0$ such that for which $C \in GL(H)$

$$A\|f\|^2 \leq \sum_{n=1}^{k}\langle f, f_n\rangle\langle Cf_n, f\rangle \leq B\|f\|^2 \tag{1}$$

for all $n \in \mathbb{N}$ and f in a dense subset V of $\mathcal{H}$. Then $\{f_n\}_{n=1}^{\infty}$ is C-controlled frame for H.

Proof. Let $g \in H = \overline{V}(-V)$ such that $A\|g\|^2 > \sum_{n=1}^{k}\langle g, f_n\rangle\langle Cf_n, g\rangle$, Thus, by density of V in $\mathcal{H}$ there exists $h \in V$ such that $\sum_{n=1}^{k}\langle h, f_n\rangle\langle Cf_n, h\rangle < A\|h\|^2$. This is contradiction and shows that the left hand of (1) holds for all elements in $\mathcal{H}$. The right hand is proved as the same way. Hence, for all $f \in \mathcal{H}$ and $n \in \mathbb{N}$ we have the (1), that shows the sequence of partial Sum below

$$\left\{ \sum_{n=1}^{k} \langle f, f_n \rangle \langle C f_n, f \rangle \right\}_{k\in\mathbb{N}},$$

is a positive, bounded sequence. Therefore, $A\|f\|^2 \leq \sum_{n=1}^{\infty} \langle f, f_n \rangle \langle C f_n, f \rangle \leq B\|f\|^2$, which means the proof is complete. □

In the following [Lemma 5.1.10 [4]], that is in fact Exercise 5.4, is proved.

Proposition 2.3. *if $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for Hilbert space $\mathcal{H}$ and $\{f_k\}_{k=1}^{\infty}$ is a sequence in this space, then we define*

$$S := \left\{ \{c_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N}) : \text{ is a finite sequence } \{c_n\}_{n=1}^{\infty} \text{ and} \sum_{n=1}^{\infty} |c_n|^2 = 1 \right\}.$$

If there exists A and $B > 0$ such that for all $\{c_n\}_{n=1}^{\infty} \in S$ we have $A \leq \sum_{n=1}^{\infty} \left| \sum_{j=1}^{\infty} c_j \langle e_j, f_n \rangle \right|^2 \leq B$, Then $\{f_n\}_{n=1}^{\infty}$ is a frame for $\mathcal{H}$ with bounds A and B.

Proof. Let $\Omega := \{f \in \mathcal{H} : \ \|f\| = 1\}$, and $D = \left\{ \sum_{j=1}^{\infty} c_j e_j : \ \{c_j\}_{j=1}^{\infty} \in S \right\}$. By the definition of S it is clear that $D \subseteq \Omega$. Now we show that D is dense in Ω. Based on [Theorem 4.18, [7]], for every $f \in \Omega$ we have $f = \sum_{j=1}^{\infty} \langle f, e_j \rangle e_j$ and $\sum_{j=1}^{\infty} |\langle f, e_j \rangle|^2 = \|f\| = 1$. If we let $g_n := \dfrac{\sum_{j=1}^{n} \langle f, e_j \rangle e_j}{\left(\sum_{j=1}^{n} |\langle f, e_j \rangle|^2 \right)^{\frac{1}{2}}}$, for every $n \in \mathbb{N}$. Then by simple calculation we can show $\{g_n\}_{n=1}^{\infty} \subseteq D$. Moreover, because of tendency of $\{g_n\}_{n=1}^{\infty}$ to f, we have density of D in Ω. Now we show that inequality of frame holds for D. for every, $f = \sum_{j=1}^{\infty} c_j e_j$ in D, we have $A\|f\|^2 = A \leq \sum_{n=1}^{\infty} \left| \sum_{j=1}^{\infty} c_j \langle e_j, f_n \rangle \right|^2 \leq B = B\|f\|^2$. Since $\sum_{n=1}^{\infty} \left| \sum_{j=1}^{\infty} c_j \langle e_j, f_n \rangle \right|^2 = \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2$, then, for every $f \in D$

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \leq B\|f\|^2. \tag{2}$$

Thus, (2) holds for every $f \in \Omega$ Now let $f \in \mathcal{H}$ is not zero. Hence $\dfrac{f}{\|f\|} \in \Omega$ and by (2),

$$A \left\| \frac{f}{\|f\|} \right\|^2 \leq \sum_{n=1}^{\infty} |\langle \frac{f}{\|f\|}, f_n \rangle|^2 \leq B \left\| \frac{f}{\|f\|} \right\|^2.$$

Thus, for all $f \in \mathcal{H}$ we have $A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \leq B\|f\|^2$. □

Proposition 2.4. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in a Hilbert space $\mathcal{H}$, and $\{e_n\}_{n=1}^{\infty}$ be orthonormal basis for H. additionally there exist constants $A, B > 0$, $K \in B(H)$ and

$$S := \left\{ \{c_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N}) | \{c_n\}_{n=1}^{\infty} \ \ is \ \ finite \ \ and \ \ \sum_{n=1}^{\infty} |c_k|^2 = 1 \right\},$$

Then If $A\|K^*\|^2 \leq \sum_{n=1}^{\infty} \left| \sum_{j=1}^{\infty} c_j \langle e_j, f_n \rangle \right|^2 \leq B$ for all $\{c_n\}_{n=1}^{\infty} \in S$, $\{f_n\}_{n=1}^{\infty}$ is a K-frame for H.

Proof. By proposition 2.3 $\{f_n\}_{n=1}^{\infty}$ is a frame for $\mathcal{H}$ by bounds $A\|K\|^2$ and B. Therefore, for every $f \in \mathcal{H}$ we have, $A\|K\|^2\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2$. Since $\|K^*f\|^2 \leq \|K\|^2\|f\|^2$, then $A\|K^*f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2$. $\square$

Now we extend this proposition to controlled-frames.

Proposition 2.5. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in a Hilbert space H, and $\{e_n\}_{n=1}^{\infty}$ be orthonormal basis for H. Let

$$S := \left\{\{c_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})| \ \{c_n\}_{n=1}^{\infty} \quad is \ \ finite \ \ and \ \ \sum_{n=1}^{\infty} |c_k|^2 = 1\right\}.$$

Additionally if there exist constants $A, B > 0$, $C \in GL(H)$ such that

$$A \leq \sum_{n=1}^{\infty} \sum_{i,j=1}^{\infty} c_i\overline{c_j}\langle e_i, f_n\rangle\langle Cf_n, e_j\rangle \leq B$$

for all $\{c_n\}_{n=1}^{\infty} \in S$, Then $\{f_n\}_{n=1}^{\infty}$ is a C-controlled frame for H.

Proof. This proposition is proved by the same way as proposition 2.3. Let $\Omega := \{f \in \mathcal{H} : \quad \|f\| = 1\}$ and $D = \left\{\sum_{j=1}^{\infty} c_je_j : \quad \{c_j\}_{j=1}^{\infty} \in S\right\}$. Hence by the proof of proposition 2.3, D is dense in Ω. Now we show that the controlled frame inequality holds for D. By some simple calculation we can see for every $f = \sum_{j=1}^{\infty} c_je_j$ in D we have

$$\sum_{n=1}^{\infty} \sum_{i,j=1}^{\infty} c_i\overline{c_j}\langle e_i, f_n\rangle\langle Cf_n, e_j\rangle = \sum_{n=1}^{\infty}\langle f, f_n\rangle\langle Cf_n, f\rangle.$$

By the assumption, for every $f \in D$ we have

$$A = A\|f\|^2 \leq \sum_{n=1}^{\infty}\langle f, f_n\rangle\langle Cf_n, f\rangle \leq B = B\|f\|^2. \tag{3}$$

Since D is dense in Ω, then (3) is valid for all $f \in \Omega$. Now let $f \in \mathcal{H}$ be nonzero, hence, $\dfrac{f}{\|f\|} \in \Omega$ and by (3), $A \leq \sum_{n=1}^{\infty}\langle \dfrac{f}{\|f\|}, f_n\rangle\langle Cf_n, \dfrac{f}{\|f\|}\rangle \leq B$. Therefore, easily we can see the proof is complete. $\square$

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